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General deformed KdV equations and their exact solutions

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Abstract. We introduce the general deformed differential operator. Using it the general deformed KdV hierarchy is constructed. As an elementary example, we give the general deformed KdV equations. In the form of the general deformation parameter expansion the general deformed KdV equations generates the systems of the non-homogeneous linearized KdV equations which are exactly solvable. In order to solve the general deformed KdV hierarchy, we introduce the general deformed exponential function. By using the dressed-operator method these exact solutions can be constructed. As an elementary example, we give some exact solutions of the non-homogeneous linearized KdV equation system derived from the general deformed KdV equations.

1. Introduction

The KdV equation [1] is widely applied in Physics. Many varieties and extensions of the KdV equation exist. An example is the q-deformed KdV equation presented by Zhang [2]. A method for obtaining the exact solutions for this q-deformed KdV equation has been supplied by Wu, Zhang and Zheng [3]. We find that the usual KdV equation can undergo a special deformation, analogous to that undergone by the quantum group, and becomes a deformed KdV one, which still owns infinitely many conservation laws and yet has exact soliton-like solutions. The essence of this progress is that after the usual differential operator is replaced by a deformed differential operator, one can still obtain the Lax pair structure similar to the usual KdV hierarchy. The deformed KdV equation and its exact solutions can transform to the systems of the usual nonlinear partial differential equations and their exact solutions. This is a significant matter, because it is not an easy task to find out some sets of the usual nonlinear partial differential equations and their exact solutions.

Recently we find in a further step that the deformation undergone by the KdV hierarchy can be generalized to a more general formalism, i.e. the deformation is almost arbitrary. This general deformations of the KdV hierarchy opens up extensively the field of possible research, bringing innumerable new varieties of the KdV hierarchy. We find that, although it is difficult to give an explicit expression for the exact solutions of the general deformed KdV equations, the exact order-by-order solutions, after the general deformed KdV equations

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transform order by order into the usual systems of nonlinear partial differential equations, can be given through a new method suggested by us. The purpose of this paper is to introduce the general deformation, and to supply the rules for determining the general deformed KdV hierarchies and their exact solutions. The simplest system, i.e. the general deformed KdV equation, and the transformation from the general deformed KdV equation to its various order systems of usual nonlinear partial differential equations, as well the exact solutions of the general deformed KdV equation and its derived systems of usual nonlinear partial differential equations, are given is this paper. These exactly solvable systems of the nonlinear partial differential equations not only supply a ready-made example to test the reliability of various criteria on integrability, but also set up an exact model to test the effectiveness of the various approximate methods for solving this problem.

2. The general deformed formal pseudo-differential operator

First we have to introduce a variable transformation generated by a Q-operator [2]

$$(Qf(x)) = f(q(x,\epsilon)) \tag{2.1}$$

where $q(x,\epsilon)$ is an arbitrary function in x and ϵ but obeys $q(x,0) = x$, which can be expanded by a formal parameter ϵ, called the deformation parameter:

$$q(x,\epsilon) = x + q_1(x)\epsilon + \frac{1}{2!}q_2(x)\epsilon^2 + \frac{1}{3!}q_3(x)\epsilon^3 + \cdots. \tag{2.2}$$

The deformation parameter ϵ is not necessarily a small one, but can be an arbitrary finite number provided the above expansion is convergent. The operator Q used in [2] is a special case, $q(x,\epsilon) = qx = x + 2x\epsilon$, $q = 1 + 2\epsilon$, i.e. $q_1(x) = 2x$, $q_2(x) = q_3(x) = \cdots = 0$, we call it the *proportional deformation*, i.e. Q-deformation. The Q-operator and its inverse can be written in the operator form

$$Q = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(q(x,\epsilon) - x)^n \partial^n \tag{2.3a}$$

$$Q^{-1} = 1 + \sum_{n=1}^{\infty} (1 - Q)^n \tag{2.3b}$$

which are obviously infinite-order operators in terms of the usual differential operator $\partial = \partial/\partial x$.

After specifying the operator Q, we define the general deformed differential operator [4]

$$D = \frac{1}{q(x,\epsilon) - x}(Q - 1) \tag{2.4}$$

which is also a usual infinite-order differential operator

$$D = \partial + \sum_{n=2}^{\infty} \frac{1}{n!}(q(x,\epsilon) - x)^{n-1}\partial^n. \tag{2.5}$$

It is obvious that D goes back to the usual differential operator ∂ when ϵ approaches 0. According to the definition of D, one can prove that the general deformed Leibniz rule is

$$(D(f(x)g(x)) = (Df(x))g(x) + (Qf(x))(Dg(x)) \tag{2.6}$$

which can be expressed in operator form as

$$Df = (Qf)D + (Df). \tag{2.7}$$

Using equation (2.7) twice, one has

$$D^2 f = (Q^2 f)D^2 + ((QDf) + (DQf))\, D + (D^2 f) \tag{2.8}$$

it is notable that generally $(QDf) \neq (DQf)$. The higher-order formulae can be obtained by using (2.7) repeatedly. It is very important that we should be able to define the inverse of D,

$$D^{-1} f = (Q^{-1} f)D^{-1} - (Q^{-1}DQ^{-1} f)D^{-2} + (Q^{-1}DQ^{-1}DQ^{-1} f)D^{-2} - \cdots \tag{2.9}$$

which is the generalization of the usual case

$$\partial^{-1} f = f\partial^{-1} - (\partial f)\partial^{-2} + (\partial^2 f)\partial^{-3} - \cdots. \tag{2.10}$$

One can check that $D \cdot D^{-1} f = D^{-1} \cdot Df = f$ by using (2.7). The D can be expressed in operator form as

$$D^{-1} = \partial^{-1} + \sum_{n=1}^{\infty} \partial^{-1}(1 - D\partial^{-1})^n. \tag{2.11}$$

By having D^{-1} act on (2.9) repeatedly, we can obtain the formulae for the higher negative order case, D^{-n}. The operators D^l and D^{-k} form a closed algebra with associativity.

Now we can define a general deformed formal pseudo-differential operator

$$K = \sum_{n=-\infty}^{M} k_n(x) D^n \tag{2.12}$$

i.e. its highest order is $M \geqslant 0$, but it has negative infinite orders of the general deformed differential operator D. We further introduce the decomposition $K = K_+ + K_-$ for the operator K:

$$K_+ = \sum_{n=0}^{M} k_n(x) D^n \tag{2.13a}$$

$$K_- = \sum_{n=1}^{\infty} k_{-n}(x) D^{-n}. \tag{2.13b}$$

It is remarkable that the general deformed differential operator includes infinitely many arbitrary functions $q_1(x), q_2(x), \ldots$; therefore we can obtain very general deformations.

3. The general deformed KdV hierarchy and the general deformed KdV equation

The construction of the general deformed KdV hierarchy is similar with usual case. The key point is that the usual differential operator ∂ is replaced by the general deformed differential operator D. The Nth general deformed KdV hierarchy consists of an infinite set of the general deformed differential equations with commuting flows, where the equations are about the coefficients $V_n(x, t_p)$ $(n = 0, 1, \ldots, N-1)$ of a general deformed differential operator L of order N that has been put in the canonical form

$$L = D^N + \sum_{n=0}^{N-1} V_n(x, t_p) D^n. \tag{3.1}$$

In the algebra of the general deformed pseudo-differential operators the L has an unique Nth root $L^{1/N}$. In the Lax representation [5] the pth flow of the Nth general deformed KdV hierarchy, called the (p, N) system, is given [6] by

$$\frac{\partial}{\partial t_p} L = [(L^{p/N})_+, L] = [L, (L^{p/N})_-] \tag{3.2}$$

where the $\{t_p\}$ are called time parameters.

The simplest system of the (p, N) general deformed KdV hierarchy is certainly be the $(3, 2)$ system, which is called the system of general deformed KdV equations. Let us give this in order to illustrate the above procedure. This model is obtained by taking L to be the following general deformed differential operator of order two:

$$L = D^2 + V_1(x, t)D + V_0(x, t). \tag{3.3}$$

The formal expansion of $L^{1/2}$ in powers of D is given by

$$L^{1/2} = D + \sum_{n=0}^{\infty} W_{-n} D^{-n}. \tag{3.4}$$

Since one later needs only the first five of the coefficients W_{-n} in the general deformed KdV equations, one gives them in terms of V_1 and V_0 order by order as

$$W_0 = (1+Q)^{-1} V_1 \tag{3.5a}$$

$$W_{-1} = -(1+Q)^{-1}(-V_0 + (DW_0) + W_0^2) \tag{3.5b}$$

$$W_{-2} = -(1+Q)^{-1}(W_{-1}(Q^{-1}W_0) + (DW_{-1}) + W_0 W_{-1}) \tag{3.5c}$$

$$\begin{aligned} W_{-3} = -(1+Q)^{-1}(&-W_{-1}(Q^{-1}DQ^{-1}W_0) + W_{-2}(Q^{-2}W_0) \\ &+W_{-1}(Q^{-1}W_{-1}) + (DW_{-2}) + W_0 W_{-2}) \end{aligned} \tag{3.5d}$$

$$\begin{aligned} W_{-4} = -(1+Q)^{-1}(&-W_{-1}(Q^{-1}DQ^{-1}DQ^{-1}W_0) - W_{-2}(Q^{-1}DQ^{-2}W_0) \\ &-W_{-2}(Q^{-2}DQ^{-1}W_0) + W_{-3}(Q^{-3}W_0) - W_{-1}(Q^{-1}DQ^{-1}W_{-1}) \\ &+W_{-2}(Q^{-2}W_{-1}) + W_{-1}(Q^{-1}W_{-2}) + (DW_{-3}) + W_0 W_{-3}). \end{aligned} \tag{3.5e}$$

The general deformed differential operator appearing in the $(3, 2)$ system is $(L^{3/2})_+$. Due to the last identity of (3.2), one needs only the first two terms of the general deformed pseudo-differential operator $(L^{3/2})_-$

$$(L^{3/2})_- = U_{-1}D^{-1} + U_{-2}D^{-2} + \cdots \tag{3.6}$$

where the coefficients U_{-1} and U_{-2} are given by the following expressions:

$$\begin{aligned} U_{-1} = &(D^2 W_{-1}) + (DQW_{-2}) + (QDW_{-2}) + (Q^2 W_{-3}) \\ &+V_1(DW_{-1}) + V_1(QW_{-2}) + V_0 W_{-1} \end{aligned} \tag{3.7a}$$

$$\begin{aligned} U_{-2} = &(D^2 W_{-2}) + (DQW_{-3}) + (QDW_{-3}) + (Q^2 W_{-4}) \\ &+V_1(DW_{-2}) + V_1(QW_{-3}) + V_0 W_{-2}. \end{aligned} \tag{3.7b}$$

Now we can obtain the general deformed KdV equations by using (3.2) (let $t_3 = t$) and (3.6):

$$\frac{\partial V_1}{\partial t} = (Q^2 U_{-1}) - U_{-1} \tag{3.8a}$$

$$\frac{\partial V_0}{\partial t} = (Q^2 U_{-2}) - U_{-2} + (DQU_{-1}) + (QDU_{-1}) + V_1(QU_{-1}) - U_{-1}(Q^{-1}V_1). \tag{3.8b}$$

It is notable that the operators D and Q play jointly important roles in the general deformed KdV equations. Not a single one of the two roles can be dispensed with. This new distinguishing feature is worthy of in-depth study.

4. The general deformed KdV equation in the deformation parameter expansion

Usually we know the conventional differential equation well and are not familiar with the general deformed differential one. We rather express the general deformed KdV equations in terms of the usual differential equations. Since it is difficult to find the general expression, we expand them only up to the second order of the deformation parameter. The operators we need are written as

$$Q = 1 + \epsilon q_1 \partial + \epsilon^2 (\tfrac{1}{2} q_2 \partial + \tfrac{1}{2} q_1^2 \partial^2) + \mathcal{O}(\epsilon^3) \tag{4.1}$$

$$Q^{-1} = 1 - \epsilon q_1 \partial + \epsilon^2 (q_1 q_1' \partial - \tfrac{1}{2} q_2 \partial + \tfrac{1}{2} q_1^2 \partial^2) + \mathcal{O}(\epsilon^3) \tag{4.2}$$

$$(1+Q)^{-1} = \tfrac{1}{2} - \epsilon \tfrac{1}{4} q_1 \partial + \epsilon^2 (-\tfrac{1}{8} q_2 \partial + \tfrac{1}{8} q_1 q_1' \partial) + \mathcal{O}(\epsilon^3) \tag{4.3}$$

$$D = \partial + \epsilon \tfrac{1}{2} q_1 \partial^2 + \epsilon^2 (\tfrac{1}{4} q_2 \partial^2 + \tfrac{1}{6} q_1^2 \partial^3) + \mathcal{O}(\epsilon^3). \tag{4.4}$$

The functions V_1 and V_0 are also expanded up to second order

$$V_1 = \epsilon G_1 + \epsilon^2 G_2 + \mathcal{O}(\epsilon^3) \tag{4.5}$$

$$V_0 = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \mathcal{O}(\epsilon^3). \tag{4.6}$$

However, we find out that the functions G_1 and G_2 are solvable, which are given by

$$G_1 = q_1 F_0 \tag{4.7a}$$

$$G_2 = q_1 F_1 + \tfrac{1}{2} q_2 F_0. \tag{4.7b}$$

Substituting them in (3.5)–(3.8) we finally obtain the general deformed KdV equations in the expanded form of order two as:

$$\partial F_0 / \partial t = \tfrac{3}{2} F_0 F_0' + \tfrac{1}{4} F_0^{(3)} \tag{4.8a}$$

$$\begin{aligned}\frac{\partial F_1}{\partial t} = {} & \tfrac{1}{4} F_1^{(3)} + \tfrac{3}{2} F_1 F_0' + \tfrac{3}{2} F_0 F_1' - \tfrac{9}{4} F_0 F_0' q_1' - \tfrac{3}{4} F_0^2 q_1'' - \tfrac{9}{8} F_0'' q_1'' \\ & - \tfrac{3}{4} q_1' F_0^{(3)} - \tfrac{5}{8} F_0' q_1^{(3)} - \tfrac{1}{8} F_0 q_1^{(4)}\end{aligned} \tag{4.8b}$$

$$\begin{aligned}\frac{\partial F_2}{\partial t} = {} & \tfrac{1}{4} F_2^{(3)} + \tfrac{3}{2} F_2 F_0' + \tfrac{3}{2} F_0 F_2' - \tfrac{9}{8} F_0^2 q_1^2 F_0' + \tfrac{3}{2} F_1 F_1' - \tfrac{3}{4} F_0^3 q_1 q_1' - \tfrac{9}{4} F_1 F_0' q_1' \\ & - \tfrac{3}{2} q_1 F_0'^2 q_1' - \tfrac{9}{4} F_0 F_1' q_1' + \tfrac{7}{8} F_0 F_0' q_1'^2 - \tfrac{9}{8} F_0 F_0' q_2' - \tfrac{3}{4} q_1^2 F_0' F_0'' \\ & - \tfrac{9}{4} F_0 q_1 q_1' F_0'' - \tfrac{3}{2} F_0 F_1 q_1'' - \tfrac{17}{8} F_0 q_1 F_0' q_1'' + \tfrac{3}{4} F_0^2 q_1' q_1'' + \tfrac{21}{8} q_1' F_0'' q_1'' \\ & - \tfrac{9}{8} F_1'' q_1'' + \tfrac{21}{16} F_0' q_1''^2 - \tfrac{3}{8} F_0^2 q_2'' - \tfrac{9}{16} F_0'' q_2'' - \tfrac{1}{2} F_0 q_1^2 F_0^{(3)} + \tfrac{17}{24} q_1'^2 F_0^{(3)} \\ & - \tfrac{3}{8} q_2' F_0^{(3)} - \tfrac{19}{24} q_1 q_1'' F_0^{(3)} - \tfrac{3}{4} q_1' F_1^{(3)} - \tfrac{1}{2} F_0^2 q_1 q_1^{(3)} - \tfrac{5}{8} F_1' q_1^{(3)} \\ & + \tfrac{23}{16} F_0' q_1' q_1^{(3)} - \tfrac{11}{16} q_1 F_0'' q_1^{(3)} + \tfrac{5}{8} F_0 q_1'' q_1^{(3)} - \tfrac{5}{16} F_0' q_2^{(3)} - \tfrac{1}{2} q_1 q_1' F_0^{(4)} \\ & - \tfrac{1}{8} F_1 q_1^{(4)} - \tfrac{5}{16} q_1 F_0' q_1^{(4)} + \tfrac{1}{4} F_0 q_1' q_1^{(4)} - \tfrac{1}{16} F_0 q_2^{(4)} \\ & - \tfrac{1}{16} q_1^2 F_0^{(5)} - \tfrac{1}{16} F_0 q_1 q_1^{(5)}.\end{aligned} \tag{4.8c}$$

If one takes the proportional deformation, $q_1 = 2x$, $q_2 = 0$, one obtains the result of [2]

$$\frac{\partial F_0}{\partial t} = \tfrac{3}{2} F_0 F_0' + \tfrac{1}{4} F_0^{(3)} \tag{4.9a}$$

$$\frac{\partial F_1}{\partial t} = \tfrac{1}{4} F_1^{(3)} + \tfrac{3}{2} F_1 F_0' + \tfrac{3}{2} F_0 F_1' - \tfrac{9}{2} F_0 F_0' - \tfrac{3}{2} F_0^{(3)} \tag{4.9b}$$

$$\frac{\partial F_2}{\partial t} = \tfrac{1}{4}F_2{}^{(3)} + \tfrac{3}{2}F_2F_0' + \tfrac{3}{2}F_0F_2' + \tfrac{3}{2}F_1F_1' - \tfrac{9}{2}F_1F_0' + \tfrac{17}{6}F_0{}^{(3)} - \tfrac{3}{2}F_1{}^{(3)}$$
$$-\tfrac{9}{2}F_0F_1' + \tfrac{7}{2}F_0F_0' - 3xF_0{}^3 - 6xF_0'^2 - 9xF_0F_0'' - 2xF_0{}^{(4)}$$
$$-\tfrac{9}{2}x^2{F_0}^2{F_0}' - 3x^2F_0'F_0'' - 2x^2F_0{F_0}^{(3)} - \tfrac{1}{4}x^2F_0{}^{(5)}. \qquad (4.9c)$$

Equations (4.8) are the expanded form of the general deformed KdV equations. Equation (4.8*a*) is just the usual KdV one. Starting with the second expanded form of the general deformed KdV equations, the general form of the equations is

$$\frac{\partial F_n}{\partial t} = \tfrac{1}{4}F_n^{(3)} + \tfrac{3}{2}F_0F_n' + \tfrac{3}{2}F_0'F_n + H_n(q_1, \ldots, q_n, F_0, \ldots, F_{n-1}). \qquad (4.10)$$

As these equations are the non-homogeneous linearized equations of the usual KdV equation, we call them the *derived non-homogeneous linearized KdV equations*. Here the nonlinear non-homogeneous terms $H_n(q_1, \ldots, q_n, F_0, \ldots, F_{n-1})$ are not arbitrary, but are determined completely by expansion of the general deformed KdV equations. One is familiar with the homogeneous linearized KdV equation [7], by the use of which the symmetries of the KdV equation can be obtained. However, we are not familiar with the non-homogeneous linearized KdV equations. Speaking generally, it is not easy to find their exact solutions. The most important property of the non-homogeneous linearized KdV equations derived from the general deformed KdV equations is their exact solvability. The method for obtaining their solution and some exact solutions of the derived non-homogeneous linearized KdV will be given in the following sections. Since the general deformation can be taken arbitrarily order by order, the derived non-homogeneous linearized KdV equations will appear in innumerable new types. This extends our field of research considerably.

5. The general deformed exponential function

The procedure for solving the general deformed KdV hierarchy is similar with the usual KdV hierarchy [8]. The difference is that the operation involved is applied to the general deformed differentials, which are more tedious than the usual ones. First we have to introduce the general deformed exponential function. Using it we construct a general deformed differential operator of order M, the so-called 'clothes' operator, which has M known null states. Finally we construct a 'dressed' operator by means of this 'clothes' operator. We can prove that this 'dressed' operator is just the exact solutions of the general deformed KdV hierarchy with M soliton-like solutions. After expanding this 'dressed' operator in the deformation parameter we obtain the exact solutions of the derived non-homogeneous linearized KdV equations. Since the deformation functions can be chosen arbitrarily, we are able to obtain many new solvable partial differential equations and their exact solutions. We believe that there are in this field many new interesting questions which merit further investigation.

In analogy with the usual case, we introduce a general deformed exponential function $\exp_q(x, a)$, the definition of which is

$$(D \exp_q(x, a)) = a \exp_q(x, a) \qquad (5.1)$$

where a is an arbitrary non-zero complex parameter. The exact form of the general deformed exponential function can be given only in special cases. For example, the proportional deformation [2] has the following results:

$$q_1(x) = 2x,\ q_2(x) = q_3(x) = \cdots = 0 \qquad (5.2a)$$

$$\exp_q(x,a) = \sum_{n=0}^{\infty} \frac{(ax)^n}{[n]!} \tag{5.2b}$$

$$[n]! = [1][2]\cdots[n] \qquad [n] = \epsilon^{-1}(1-(1-\epsilon)^n). \tag{5.2c}$$

It is not easy to find the general formulae of the general deformed exponential function in general cases. Therefore we need the expression of $\exp_q(x,a)$ expanded in terms of ϵ. Considering that (5.1) becomes

$$\partial e^{ax} = a\, e^{ax} \tag{5.3}$$

when ϵ equals zero, we suppose the expanded expression of $\exp_q(x,a)$ in ϵ is

$$\exp_q(x,a) = e^{ax}(1+\epsilon h_1(x,a)+\epsilon^2 h_2(x,a)+\cdots). \tag{5.4}$$

Using equation (4.4) and substituting (5.4) into (5.1), we obtain the system of equations satisfied by the $h_i(x,a)$ as

$$h_1' + \tfrac{1}{2}a^2 q_1 = 0 \tag{5.5a}$$

$$h_2' + \tfrac{1}{2}a^2 q_1 h_1 + a q_1 h_1' + \tfrac{1}{2} q_1 h_1'' + \tfrac{1}{6}a^3 q_1^2 + \tfrac{1}{4}a^2 q_2 = 0. \tag{5.5b}$$

Solving these equations we obtain

$$h_1(x,a) = -\tfrac{1}{2}a^2 \int q_1\, dx \tag{5.6a}$$

$$h_2(x,a) = \tfrac{1}{4}a^4 \int q_1 \left(\int q_1\, dx\right) dx + \tfrac{1}{3}a^3 \int q_1^2\, dx - \tfrac{1}{4}a^2 \int q_2\, dx + \tfrac{1}{8}a^2 q_1^2 \tag{5.6b}$$

where all constants of integration are taken to be zero for simplicity. It can be easily deduced that this process of solution can go on order by order without difficulty. Then the expanded form of the general deformed exponential function can be given.

Now let us construct the M-functions, which are very useful later on:

$$y_k(x,t_p) = \exp_q(x,a_k)\cdot \exp(a_k^p \cdot t_p) + b_k \exp_q(x,\rho a_k)\cdot \exp(\rho^p a_k^p \cdot t_p) \tag{5.7}$$

where $\{a_k\}, \{b_k\}$, $(k=1,\ldots,M)$ are arbitrary complex constants with the requirement that $a_k \neq a_l$ for $k \neq l$, and ρ is the Nth root of 1, i.e.

$$\rho^N = 1. \tag{5.8}$$

They are independent functions due to the different choice of parameters a_k. In accordance with definitions (5.1), (5.7) and (5.8), one can easily verify that the functions $y_k(x,t_p)$ have the following important property:

$$\frac{\partial}{\partial t_p} y_k = (D^p y_k) \tag{5.9}$$

$$(D^N y_k) = a_k^N \cdot y_k. \tag{5.10}$$

We need to construct a Mth-order general deformed differential operator Φ which takes the above-mentioned M-functions as its null states:

$$\Phi \cdot y_k = 0 \qquad (k=1,\ldots,M). \tag{5.11}$$

Here Φ is given by

$$\Phi = \frac{1}{\Delta_M} \det \begin{pmatrix} y_1, & \cdots, & y_M, & 1 \\ (Dy_1), & \cdots, & (Dy_M), & D \\ \vdots & \ddots & \vdots & \vdots \\ (D^M y_1), & \cdots, & (D^M y_M), & D^M \end{pmatrix}. \tag{5.12}$$

Since there are two columns to be equal in the determinant after Φ acts on y_k, it is obvious that (5.11) is proved. The above determinant expands in the last column and its minors are placed in front of the operators D^i. This means that the operator Φ has the following expansion:

$$\Phi = D^M + Z_{M-1}D^{M-1} + \cdots + Z_0 \tag{5.13}$$

$$Z_i = (-1)^{M-i}\Delta_i/\Delta_M \tag{5.14}$$

where Δ_i is the determinant of remained matrix of the following $(M+1) \times M$ matrix deleted the $(i+1)$th row elements,

$$\Delta = \begin{pmatrix} y_1, & \cdots, & y_M \\ (Dy_1), & \cdots, & (Dy_M) \\ \vdots & \ddots & \vdots \\ (D^M y_1), & \cdots, & (D^M y_M) \end{pmatrix} \tag{5.15}$$

$$\Delta_i = \det(\Delta \text{ without the } (i+1)\text{th row}) \qquad (i = M, M-1, \ldots, 1, 0). \tag{5.16}$$

Due to the independence of the M-functions y_k, the first determinant Δ_M is non-zero:

$$\Delta_M \neq 0. \tag{5.17}$$

The inverse of the operator Φ can be defined, being a general deformed pseudo-differential operator:

$$\Phi^{-1} = D^{-M} + v_1 D^{-M-1} + v_2 D^{-M-2} + \cdots. \tag{5.18}$$

In accordance with the calculating rules given in section 2, the operator Φ^{-1} can be expressed by

$$\Phi^{-1} = D^{-M}(1 - Z_{M-1}D^{-1} + (-Z_{M-2} + Z_{M-1}(Q^{-1}Z_{M-1}))D^{-2} + \cdots). \tag{5.19}$$

In the following section we will use the operator Φ as 'clothes' to 'dress' some operator.

6. The 'dressed' operator

We define a 'dressed' operator

$$L = \Phi D^N \Phi^{-1} \tag{6.1}$$

where the 'clothes' Φ are defined by (5.12). We shall prove that this 'dressed' operator is simply the solutions of the general deformed KdV hierarchy introduced in section 3:

$$\frac{\partial L}{\partial t_p} = [L_+^{p/N}, L]. \tag{6.2}$$

The method of proof is similar to that due to Dickey [8] and has been used in [3]. Since the L-operator used in the general deformed KdV hierarchy is the Nth-order general deformed differential operator, we must confirm that the L-operator defined by (6.1) does not have the negative power terms of D. To do this a decomposition $L = L_+ + L_-$ is introduced, where L_+ keeps the part of the non-negative powers of D, and L_- is its negative power part. Thus we have

$$L_+\Phi - \Phi D^N = -L_-\Phi \tag{6.3}$$

and then the two sides act on the M-functions y_k given by (5.7) as

$$(L_-\Phi)y_k = -(L_+\Phi)y_k + (\Phi D^N)y_k = -L_+(\Phi y_k) + \Phi(D^N y_k)$$
$$= -L_+ \cdot 0 + \Phi \cdot a_k^N y_k = a_k^N \cdot \Phi y_k = 0. \tag{6.4}$$

Equation (5.10) is used here. It is notable that we only have $L_+ \cdot 0 = 0$ and we *cannot* make the deduction $(L_-\Phi)y_k = L_-(\Phi y_k) = L_- \cdot 0 = 0$ due to $L_- \cdot 0 \neq 0$. From the r.h.s. of (6.3), the highest order of the operator D in L must be $(M-1)$. From the l.h.s. of (6.3) the lowest order of the operator D in L has to be zero. Therefore the operator $L_-\Phi$ is the $(M-1)$th-order general deformed differential operator. From $\Delta_M \neq 0$ and $(L_-\Phi)y_k = 0$ we obtain for this $(M-1)$th-order operator:

$$L_-\Phi = 0. \tag{6.5}$$

Due to the existence of the inverse of Φ, we have

$$L_- = 0 \cdot \Phi^{-1} = 0. \tag{6.6}$$

Therefore the operator L constructed by (6.1) is indeed the Nth-order general deformed differential operator.

Using the definition of the 'dressed' operator, it is easy to give its Nth root,

$$L^{1/N} = \Phi D \Phi^{-1}. \tag{6.7}$$

The operator $L^{p/N}$ is also easily obtained to be

$$L^{p/N} = \Phi D^p \Phi^{-1}. \tag{6.8}$$

Let

$$R_p = L_+^{p/N} \tag{6.9a}$$

$$S = R_p\Phi - \Phi D^p \tag{6.9b}$$

we have

$$S = -L_-^{p/N}\Phi. \tag{6.10}$$

We obtain the derivative of (5.11) with respect to t_p:

$$0 = \dot{\Phi} y_k + \Phi \dot{y}_k = \dot{\Phi} y_k + \Phi D^p y_k = \dot{\Phi} y_k + R_p \Phi y_k - S y_k = (\dot{\Phi} - S) y_k \tag{6.11}$$

where (5.9) is used. From (6.10) the highest order of the operator S is $(M-1)$, and from (6.9a) the lowest order of S is zero. So is $\dot{\Phi}$ due to the highest term of Φ being independent of time. Therefore the operator $(\dot{\Phi} - S)$ is the $(M-1)$th-order general deformed differential operator. Similar with the reason that the operator $L_-\Phi$ equals zero, i.e. $\Delta_M \neq 0$ and $(\dot{\Phi} - S)y_k = 0$, we obtain

$$\dot{\Phi} - S = 0 \qquad \text{i.e. } \dot{\Phi} = R_p\Phi - \Phi D^p. \tag{6.12}$$

Now let us consider the time evolution of the 'dressed' operator:

$$\begin{aligned} \dot{L} &= \dot{\Phi} D^N \Phi^{-1} - \Phi D^N \Phi^{-1} \dot{\Phi} \Phi^{-1} \\ &= (R_p\Phi - \Phi D^p) D^N \Phi^{-1} - \Phi D^N \Phi^{-1} (R_p \Phi - \Phi D^p) \Phi^{-1} \\ &= R_p \Phi D^N \Phi^{-1} - \Phi D^N \Phi^{-1} R_p \\ &= R_p L - L R_p \end{aligned} \tag{6.13}$$

i.e. $\dot{L} = [L_+^{p/N}, L]$ is established.

In conclusion, therefore, the 'dressed' operator (6.1) automatically gives the solutions of the general deformed KdV hierarchy. Since Φ is given by (5.12), the y_k by (5.7), and the $\exp_q(x, a)$ by (5.4), (5.6), we can finally obtain the concrete expression of each coefficient of the operator L, i.e. the exact solutions of the general deformed differential equations.

7. The exact solutions of the non-homogeneous linearized KdV equations

As a standard example of the general deformed KdV hierarchy, we discuss the general deformed KdV equations and their non-homogeneous linearized equations. In this case $p=3, N=2$ and $\rho=-1$. In accordance with the notation of sections 3 and 4,

$$L = D^2 + V_1 D + V_0 \tag{3.3}$$

where

$$V_0 = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \cdots \tag{4.6}$$

$$V_1 = \epsilon q_1 F_0 + \epsilon^2 (q_1 F_1 + \tfrac{1}{2} q_2 F_0) + \cdots. \tag{4.5,7}$$

What we want to get are the M soliton-like solutions, where M is the soliton number. Using (5.13) and (5.19) in $L = \Phi D^2 \Phi^{-1}$, we obtain

$$V_1 = Z_{M-1} - (Q^2 Z_{M-1}) \tag{7.1}$$

$$V_0 = Z_{M-2} - (Q^2 Z_{M-2}) - (DQZ_{M-1}) - (QDZ_{M-1}) \\ -(QZ_{M-1})(Z_{M-1} - (Q^2 Z_{M-1})) \tag{7.2}$$

where

$$Z_{M-1} = -\frac{\Delta_{M-1}}{\Delta_M} \qquad Z_{M-2} = \frac{\Delta_{M-2}}{\Delta_M}. \tag{7.3}$$

However, the general expressions of Z_i are too complicated to be written; thus we only consider the double soliton-like case for simplicity. In this case, $M=2$, we take a_1, b_1, a_2 and b_2 as the double soliton-like parameters, and obtain y_1 and y_2 as:

$$\begin{aligned}
y_k(x,t) = {} & \big(b_k \exp(-a_k{}^3 t - a_k x) + \exp(a_k{}^3 t + a_k x)\big) \\
& + \Big[-\tfrac{1}{2} a_k{}^2 b_k \exp(-a_k{}^3 t - a_k x) \left(\int q_1 \, \mathrm{d}x\right) \\
& - \tfrac{1}{2} a_k{}^2 \exp(a_k{}^3 t + a_k x) \left(\int q_1 \, \mathrm{d}x\right) \Big] \epsilon \\
& + \Big[\tfrac{1}{8} a_k{}^4 b_k \exp(-a_k{}^3 t - a_k x) \left(\int q_1 \, \mathrm{d}x\right)^2 \\
& + \tfrac{1}{8} a_k{}^4 \exp(a_k{}^3 t + a_k x) \left(\int q_1 \, \mathrm{d}x\right)^2 \\
& - \tfrac{1}{3} a_k{}^3 b_k \exp(-a_k{}^3 t - a_k x) \left(\int q_1{}^2 \, \mathrm{d}x\right) \\
& + \tfrac{1}{3} a_k{}^3 \exp(a_k{}^3 t + a_k x) \left(\int q_1{}^2 \, \mathrm{d}x\right) \\
& - \tfrac{1}{4} a_k{}^2 b_k \exp(-a_k{}^3 t - a_k x) \left(\int q_2 \, \mathrm{d}x\right) \\
& - \tfrac{1}{4} a_k{}^2 \exp(a_k{}^3 t + a_k x) \left(\int q_2 \, \mathrm{d}x\right)
\end{aligned}$$

$$+\frac{1}{8}a_k{}^2b_k\exp(-a_k{}^3t-a_kx)q_1{}^2+\frac{1}{8}a_k{}^2\exp(a_k{}^3t+a_kx)q_1{}^2\Big]\epsilon^2$$

$$+\mathrm{O}(\epsilon)^3. \tag{7.4}$$

The Δ_2, Δ_1 and Δ_0 are given by

$$\Delta_2=\det\begin{pmatrix} y_1, & y_2 \\ (Dy_1), & (Dy_2)\end{pmatrix} \tag{7.5a}$$

$$\Delta_1=\det\begin{pmatrix} y_1, & y_2 \\ (D^2y_1), & (D^2y_2)\end{pmatrix} \tag{7.5b}$$

and

$$\Delta_0=\det\begin{pmatrix} (Dy_1), & (Dy_2) \\ (D^2y_1), & (D^2y_2)\end{pmatrix}. \tag{7.5c}$$

Using all of these expressions we can obtain the functions F_0, F_1 and F_2 by means of (7.2)–(7.5). Since the general formulae keeping undetermined deformation are so long that they are not suitable to be published, we only give a special case as an example. In order to do this we take a concrete general deformation at will:

$$q_1=x^3 \qquad q_2=2x. \tag{7.6}$$

Then the expressions for F_0, F_1 and F_2 given in (4.8) become

$$\dot F_0=\frac{3}{2}F_0F_0'+\frac{1}{4}F_0{}^{(3)}, \tag{7.7a}$$

$$\dot F_1=\frac{1}{4}F_1{}^{(3)}+\frac{3}{2}F_0F_1'+\frac{3}{2}F_1F_0'-\frac{9}{2}xF_0{}^2-\frac{15}{4}F_0'-\frac{27}{4}x^2F_0F_0'-\frac{27}{4}xF_0''-\frac{9}{4}x^2F_0{}^{(3)} \tag{7.7b}$$

$$\begin{aligned}\dot F_2=&\frac{1}{4}F_2{}^{(3)}+\frac{3}{2}F_0F_2'+\frac{3}{2}F_2F_0'+\frac{45}{2}xF_0+\frac{21}{2}x^3F_0{}^2-\frac{9}{4}x^5F_0{}^3-9xF_0F_1\\&+\frac{585}{8}x^2F_0'-\frac{9}{4}F_0F_0'-\frac{39}{8}x^4F_0F_0'-\frac{9}{8}x^6F_0{}^2F_0'-\frac{27}{4}x^2F_1F_0'-\frac{9}{2}x^5F_0'^2\\&-\frac{15}{4}F_1'-\frac{27}{4}x^2F_0F_1'+\frac{3}{2}F_1F_1'+\frac{345}{8}x^3F_0''-\frac{27}{4}x^5F_0F_0''-\frac{3}{4}x^6F_0'F_0''\\&-\frac{27}{4}xF_1''-\frac{3}{4}F_0{}^{(3)}+\frac{13}{8}x^4F_0{}^{(3)}-\frac{1}{2}x^6F_0F_0{}^{(3)}-\frac{9}{4}x^2F_1{}^{(3)}\\&-\frac{3}{2}x^5F_0{}^{(4)}-\frac{1}{16}x^6F_0{}^{(5)}.\end{aligned} \tag{7.7c}$$

The double soliton-like parameters have to be taken as concrete values at will in order to avoid long results:

$$a_1=1 \qquad b_1=2 \qquad a_2=2 \qquad b_2=1. \tag{7.8}$$

Then from (7.4) we have:

$$\begin{aligned}y_1=&(2\mathrm{e}^{-t-x}+\mathrm{e}^{t+x})+x^4(-\tfrac{1}{4}\mathrm{e}^{-t-x}-\tfrac{1}{8}\mathrm{e}^{t+x})\epsilon\\&+[(-\tfrac{1}{2}x^2+\tfrac{1}{4}x^6-\tfrac{2}{21}x^7+\tfrac{1}{64}x^8)\mathrm{e}^{-t-x}\\&+(-\tfrac{1}{4}x^2+\tfrac{1}{8}x^6+\tfrac{1}{21}x^7+\tfrac{1}{128}x^8)\mathrm{e}^{t+x}]\epsilon^2+\mathrm{O}(\epsilon)^3\end{aligned} \tag{7.9a}$$

$$\begin{aligned}y_2=&(\mathrm{e}^{-8t-2x}+\mathrm{e}^{8t+2x})+x^4(-\tfrac{1}{2}\mathrm{e}^{-8t-2x}-\tfrac{1}{2}\mathrm{e}^{8t+2x})\epsilon\\&+\Big[(-x^2+\tfrac{1}{2}x^6-\tfrac{8}{21}x^7+\tfrac{1}{8}x^8)\mathrm{e}^{-8t-2x}\\&+(-x^2+\tfrac{1}{2}x^6+\tfrac{8}{21}x^7+\tfrac{1}{8}x^8)\mathrm{e}^{8t+2x}\Big]\epsilon^2+\mathrm{O}(\epsilon)^3.\end{aligned} \tag{7.9b}$$

Finally, from (7.2) and (4.6) the exact solution of (7.7*a*), (7.7*b*) is given by

$$
\begin{aligned}
F_0 = \tfrac{8}{7}\Big(&168e^{2t+2x} + 504e^{4t+4x} - 1344e^{16t+4x} + 378e^{6t+6x} - 6048e^{18t+6x} \\
&-8064e^{20t+8x} + 8064e^{32t+8x} - 3528e^{22t+10x} + 21168e^{34t+10x} - 756e^{24t+12x} \\
&+15120e^{36t+12x} - 12096e^{48t+12x} + 5292e^{38t+14x} - 14112e^{50t+14x} \\
&+504e^{40t+16x} - 8064e^{52t+16x} - 1512e^{54t+18x} + 1512e^{66t+18x} \\
&-84e^{56t+20x} + 504e^{68t+20x} + 42e^{70t+22x}\Big) \\
&\times \left(-2 - 3e^{2t+2x} + 6e^{16t+4x} + e^{18t+6x}\right)^{-4} \qquad (7.10a)
\end{aligned}
$$

$$
\begin{aligned}
F_1 = 24x^2\Big[&-(6+8x)e^{2t+2x} + (-9+12x)e^{4t+4x} + (48+128x)e^{16t+4x} \\
&+(126+72x)e^{18t+6x} + (69+116x)e^{20t+8x} + (-144+384x)e^{32t+8x} \\
&+(18+48x)e^{22t+10x} + (-138+232x)e^{34t+10x} + (-63+36x)e^{36t+12x} \\
&+(-6+16x)e^{38t+14x} + (18+24x)e^{50t+14x} + (3-4x)e^{52t+16x}\Big] \\
&\times \left(-2 - 3e^{2t+2x} + 6e^{16t+4x} + e^{18t+6x}\right)^{-3} \qquad (7.10b)
\end{aligned}
$$

$$
\begin{aligned}
F_2 = \tfrac{8}{7}\Big[&(252 + 756x^4 - 2520x^6)e^{4t+4x} + (4032 + 12096x^4 - 161280x^6)e^{32t+8x} \\
&+(7560 + 22680x^4 - 79632x^6)e^{36t+12x} + (252 + 756x^4 - 10080x^6)e^{40t+16x} \\
&+(252 + 756x^4 - 2520x^6)e^{68t+20x} \\
&+(84 + 336x + 252x^4 + 1008x^5 + 504x^6 + 16x^7)e^{2t+2x} \\
&+(-672 - 5376x - 2016x^4 - 16128x^5 - 16128x^6 - 1024x^7)e^{16t+4x} \\
&+(189 - 756x + 567x^4 - 2268x^5 + 1134x^6 - 36x^7)e^{6t+6x} \\
&+(-3024 - 12096x - 9072x^4 - 36288x^5 - 2016x^6 - 2880x^7)e^{18t+6x} \\
&+(-4032 - 8064x - 12096x^4 - 24192x^5 - 42336x^6 - 3072x^7)e^{20t+8x} \\
&+(-1764 - 9072x - 5292x^4 - 27216x^5 - 30744x^6 - 2160x^7)e^{22t+10x} \\
&+(10584 - 22176x + 31752x^4 - 66528x^5 - 130032x^6 - 2592x^7)e^{34t+10x} \\
&+(-378 - 3024x - 1134x^4 - 9072x^5 - 9072x^6 - 576x^7)e^{24t+12x} \\
&+(-6048 + 48384x - 18144x^4 + 145152x^5 - 145152x^6 + 9216x^7)e^{48t+12x} \\
&+(2646 + 5544x + 7938x^4 + 16632x^5 - 32508x^6 + 648x^7)e^{38t+14x} \\
&+(-7056 + 36288x - 21168x^4 + 108864x^5 - 122976x^6 + 8640x^7)e^{50t+14x} \\
&+(-4032 + 8064x - 12096x^4 + 24192x^5 - 42336x^6 + 3072x^7)e^{52t+16x} \\
&+(-756 + 3024x - 2268x^4 + 9072x^5 - 504x^6 + 720x^7)e^{54t+18x} \\
&+(756 + 3024x + 2268x^4 + 9072x^5 + 4536x^6 + 144x^7)e^{66t+18x} \\
&+(-42 + 336x - 126x^4 + 1008x^5 - 1008x^6 + 64x^7)e^{56t+20x} \\
&+(21 - 84x + 63x^4 - 252x^5 + 126x^6 - 4x^7)e^{70t+22x}\Big] \\
&\times \left(-2 - 3e^{2t+2x} + 6e^{16t+4x} + e^{18t+6x}\right)^{-4} \qquad (7.10c)
\end{aligned}
$$

where F_0 is a standard double soliton solution of the usual KdV equation (7.7*a*). Due to this reason we call the solutions F_0, F_1 and F_2 double soliton-like solutions. Substituting solutions (7.10) into their equations (7.7) one can check directly that they are indeed the exact solutions of these equations. In order for reader to do this check more easily, we are able to supply by e-mail a Mathematica program which can save keyboarding and check that these functions are certainly the exact solution of these equations. This check confirms again that our proof and conclusion in section 6 is correct. Of course, we can produce infinitely many sets of exact solutions for same equations in accordance with the procedure described before. Only due to space limitations do we give this set of special exact solution as an example.

It is remarkable that only the exact solutions, and not the approximate ones, of the non-homogeneous linearized KdV equations are obtained through the expansion of the deformation parameter ϵ. The function V_0 seems to be the generating functional of the exact solutions.

8. Conclusion

By using the expansion method in the deformation parameter, we obtain the approximate equations of the general deformed KdV equation order by order, i.e. the derived non-homogeneous linearized KdV equations. By using the same method we obtain the approximate solutions of the general deformed KdV equation order by order. However, the approximate solutions of various orders are just the exact solutions of the approximate equations of the various orders of the general deformed KdV equation! The reason is simple. They are only the different expanded forms of the just same identity. However, from the point of view of nonlinear partial differential equation theory, this should be a new phenomenon about the exact solutions which is worthy of in-depth study.

It is clear that this solving procedure not only can find the exact solutions with the arbitrary soliton number for the general deformed KdV equations with the general deformation chosen concretely (if your computer has enough memory), but also can obtain the exact solutions of the higher cases, for example, the general deformed Boussinesq equations and their derived non-homogeneous linearized Boussinesq equations.

It is notable that the whole method of research, from the general deformed KdV hierarchy to their exact solutions, can be generalized to the various extensions of KdV hierarchy, such as the KP hierarchy, supersymmetry, etc. The reason is essentially that if the usual differential operator is replaced by the general deformed differential operator in the usual KP hierarchy [9], then the general deformed KP hierarchy can be obtained. This work will be published by us elsewhere.

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